A model for efficient dynamical ranking in networks

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We present a physics-inspired method for inferring dynamic rankings in directed temporal networks networks in which each directed and timestamped edge reflects the outcome and timing of a pairwise interaction. The inferred ranking of each node is real-valued and varies in time as each new edge, encoding an outcome like a win or loss, raises or lowers the node's estimated strength or prestige, as is often observed in real scenarios including sequences of games, tournaments, or interactions in animal hierarchies. Our method works by solving a linear system of equations and requires only one parameter to be tuned. As a result, the corresponding algorithm is scalable and efficient. We test our method by evaluating its ability to predict interactions (edges' existence) and their outcomes (edges' directions) in a variety of applications, including both synthetic and real data. Our analysis shows that in many cases our method's performance is better than existing methods for predicting dynamic rankings and interaction outcomes.

I. Introduction

When considering a collection of people, animals, teams, or other entities, there is often an underlying hierarchy structuring the system. This hierarchy may be formally instilled in the sense that some individuals are explicitly granted certain ranks based on positions of authority. For example, in a school, there are students, teachers, and the principal or head of the school, with each position explicitly known and ranked in terms of level of authority. Alternatively, a hierarchy may be implicit in the sense that the ranks are not explicitly granted or known, but instead encoded in behaviors or interactions. For example, in animal dominance hierarchies, animals may be preferentially aggressive toward those lower in rank. In both explicit and implicit cases, hierarchies can be determined by analyzing the patterns of interactions between the entities of the sys-

If we wish to infer the ranks of entities in a hierarchical structure from the patterns of their interactions, we can treat ranks as either static or dynamic, and as ordinal or real-valued.

In the static case, time is irrelevant, and we treat all the interactions at once regardless of the sequence in which they occur, as one might when ranking the teams in a sports league at the end of a seasons.

In the dynamic case, each individual's ranking may rise or fall over time, retaining the memory of past interactions while taking new interactions into account. This can be seen in leagues such as the U.S. National Basketball Association (NBA) where rankings derived from recent games provide insight for predicting games in the near future, yet the rankings themselves may nevertheless change slowly over the course of a season or seasons. We are also interested in real-valued ranks, rather than ordinal ranks, such that the size of rank difference between two entities is an interpretable and predictive quantity, regardless of whether they are adjacent or well separated in ordinal rank.

To model systems of this type we propose Dynamical SpringRank. This builds on the previouslyproposed SpringRank algorithm [1] by incorporating time information, inferring a dynamic hierarchy from a dynamic network: that is, a dataset of timestamped interactions. each of which defines a directed edge $i \rightarrow j$ indicating that i "beat" j at time t. We make similar physically-inspired assumptions as SpringRank, modeling directed edges as springs and assuming that entities are more likely to interact if their ranks are not too far apart. We also propose a generative model for constructing directed, hierarchical networks that evolve over time.

Finally, we evaluate Dynamical SpringRank on a variety of synthetic and real datasets. From our findings, we conclude that it accurately and efficiently infers ranks and predicts the direction of edges in dynamic settings. Furthermore, it frequently outperforms other algorithms such as the Elo Rating System and Whole-History Rating.

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II. Related Work

Estimating hidden hierarchies from pairwise interactions is a fundamental problem in a wide variety of contexts. Several models have been proposed to study static hierarchies,

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i.e., scenarios where ranks do not change in time: spectral methods including Eigenvector Centrality [2], PageRank [3] and Rank Centrality [4]; approaches that target ordinal rankings, such as Minimum Violation Rank [5–7], Ranked Stochastic Block Model [8], SerialRank [9] and SyncRank [10]; Random Utility Models [11] such as the Bradley-Terry-Luce (BTL) model [12, 13]; fully generative models including the Probabilistic Niche Model of ecology [14–16]; models of friendship based on social status [17] or with hierarchy and community structure [18]; latent space models [19]; and physics-inspired models such as SpringRank [1] and belief propagation in continuous spin systems [20].

In contrast, online methods, which update ranks after each interaction, model dynamic environments where ranks vary in time and interactions have a relevant chronological order. For instance, the Elo Rating System [21], commonly used for rating chess players, is one of the most popular online methods. It was later improved by the Glicko system [22], which incorporates a measure of reliability in estimating ranks to capture their uncertainty due to, for instance, a period of inactivity or lack of data. Another approach is a win-loss ranking algorithm [23] and its dynamic extension [24]. A Bayesian ranking system inferring individual ranks from team-level outcomes is the so called TrueSkill algorithm [25], which can be seen as a generalization of the Elo system. This has been extended by TrueSkill Through Time (TTT) [26] which infers smooth time series of ranks. Decaying-history ratings such as [24] act directly on the data observations, progressively forgetting old interactions. One drawback of this approach is that time decay increases the uncertainty of player ratings: players who stop playing for a while may experience huge jumps in their ratings when they start playing again. On the other hand, players who play very frequently may have the feeling that their rating is stuck. If players do not all play at the same frequency, there is no clear way to tune the decay rate [27].

Finally, a third set of *offline* methods treats ranks as time-varying, but infers the ranks at each time-step by considering the totality of all observations, including those before and after any particular time step. For instance, the Whole-History Rating (WHR) [27], a Bayesian approach based on the dynamic Bradley-Terry-Luce model, computes the exact maximum a posteriori estimate of ranks over the whole history of all players.

III. The Model

We represent a series of interactions between N individuals as a sequence of weighted directed networks with adjacency matrix A^t for $t=0,1,2,\ldots,T$. For each t, its entry A^t_{ij} is the outcome of interactions $i \to j$ suggesting that i is ranked above j. This allows both cardinal and ordinal inputs. For instance, in team sports, A^t_{ij} could be the number of points by which team i beat team j, or we could simply set $A^t_{ij} = 1$ to indicate that i won and j lost. We can include the case where individuals interact multiple times at time t

by summing the corresponding entries.

We assume that the values of A_{ij}^t are influenced by a vector of real-valued ranks $\mathbf{s}^t = (s_1^t, \dots, s_N^t)$, where s_i^t is i's skill, strength or prestige at time t. To model these interactions, we follow SpringRank's approach of imagining the network as a physical system [1]. Specifically, each node i is embedded in $\mathbb R$ at position s_i^t , and each directed edge $i \to j$ becomes an oriented spring with a non-zero resting length and displacement $s_i^t - s_j^t$. Since we are free to rescale latent space and the energy scale, we set the spring constant and resting length to 1. The spring corresponding to an edge $i \to j$ at time t then has energy

$$H_{ij}(s_i^t, s_j^t) = \frac{1}{2} \left(s_i^t - s_j^t - 1 \right)^2.$$
 (1)

If there were no other effects, the total energy of the system at time *t* would then be

$$H^{t}(\mathbf{s}^{t}) = \sum_{i,j=1}^{N} A_{ij}^{t} H_{ij}(s_{i}^{t}, s_{j}^{t}).$$
 (2)

If we determined \mathbf{s}^t by minimizing H^t for each t separately, we would simply be applying the static SpringRank model separately to each "snapshot" of the network. This would ignore all previous (and future) interactions, and ignore the hypothesis that ranks change smoothly from one time-step to the next.

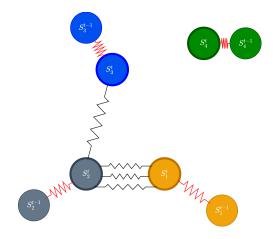


FIG. 1. A visual representation of Dynamical SpringRank. Each node i has rank s_i at time t and each edge is represented as a spring. The red springs indicate self-springs that connect past and present ranks. The black springs indicate interactions with different entities. The blue and grey nodes interact once while the grey and gold nodes interact three times. In contrast, the green node does not interact with the other entities.

To model this smoothness, we also assume a dependence between ranks at successive time-steps. Specifically, we extend the Hamiltonian (2) with an extra term that models the *self-interaction* between past and current ranks,

$$H_{\text{self}}^{t}(\mathbf{s}^{t}, \mathbf{s}^{t-1}) = \frac{k}{2} \sum_{i=1}^{N} (s_{i}^{t} - s_{i}^{t-1})^{2}.$$
 (3)

This can be seen as a set of additional "self-springs" that connect the rank of each individual with its own previous rank. The spring constant k parametrizes how smoothly we want the ranks to change from one step to the next. In inference terms, k is a hyperparameter which we tune using cross-validation.

Summing over all time-steps $0 < t \le T$ and adding this to the pairwise interactions at each time-step then gives a total energy

$$H_{\text{total}}(\{\mathbf{s}^t\}) = \sum_{t=0}^{T} H^t(\mathbf{s}^t) + \sum_{t=1}^{T} H^t_{\text{self}}(\mathbf{s}^t, \mathbf{s}^{t-1}).$$
 (4)

We call this the dynamical SpringRank Hamiltonian. The optimal ranks $\mathbf{s}^0, \mathbf{s}^1, \dots, \mathbf{s}^T$ are those that minimize it.

There are two ways to minimize H_{total} . One is to proceed in an online way, moving forward in time. In this approach, we use the static SpringRank model Eq. (2) to find the initial ranks \mathbf{s}^0 by minimizing $H^0(\mathbf{s}^0)$. As in Ref. [1], the energy is unchanged if we add a constant to all the ranks; we can break this translational symmetry by setting the mean initial rank $(1/N)\sum_{i=1}^N v_i^0$ to zero. Then, at each subsequent time-step $t \geq 1$, we update the ranks by taking into account both the new pairwise interactions and the self-springs connecting the ranks with their previous values. Namely, given \mathbf{s}^{t-1} and A^t , we find the ranks \mathbf{s}^t that minimize $H^t(\mathbf{s}^t) + H^t_{\text{self}}(\mathbf{s}^t, \mathbf{s}^{t-1})$.

Since this is a convex function of \mathbf{s}^t , we can find its minimum by setting its gradient to zero, or equivalently by balancing all the forces v_i^t . This yields a system of linear equations:

$$\left[D^{out,t} + D^{in,t} - \left(A^t + (A^t)^{\dagger}\right) + k\mathbb{I}\right] \mathbf{s}^t = \left[D^{out,t} - D^{in,t}\right] \mathbf{1}
+ k \mathbf{s}^{t-1}.$$
(5)

Here $D^{out,t}$ and $D^{in,t}$ are diagonal matrices whose entries are the weighted out- and in-degrees $D^{out,t}_{ii} = \sum_j A^t_{ij}$ and $D^{in,t}_{ii} = \sum_j A^t_{ji}$; † denotes the transpose; \mathbb{I} is the identity matrix; and $\mathbf{1}$ is the all-ones vector.

The matrix on the left side of Eq. (5) is invertible if k > 0. In particular, its eigenvector **1** has eigenvalue Nk. Thus for each A^t and each \mathbf{s}^{t-1} , Eq. (5) has a unique solution \mathbf{s}^t . Overall, Eq. (5) is similar to the regularized version of SpringRank [1] with regularization parameter $\alpha = k$. However, unlike the static model, there is a term on the right-hand side containing the previous ranks \mathbf{s}^{t-1} , creating a Markovian dependence between successive time-steps. We refer to this model as Dynamical SpringRank (DSR).

Importantly the online DSR approach does not actually minimize H_{total} , instead solving a sequence of minimization problems, one for each time step. To minimize H_{total} instead, we set $\nabla H_{\text{total}}(s^t) = 0$, solving for the minimizers s^t over all N(T+1) ranks simultaneously, yielding the following system of equations (SI Sec. S2):

$$\left[D^{out,t} + D^{in,t} - \left(A^t + \left(A^t\right)^{\dagger}\right) + 2k\mathbb{I}\right] \mathbf{s}^t = \left[D^{out,t} - D^{in,t}\right] \mathbf{1} \\
+ k \left(\mathbf{s}^{t-1} + \mathbf{s}^{t+1}\right).$$
(6)

This differs from Eq. (4) in that the right-hand side now includes both past and future ranks (which doubles the contribution of k on the left). We remove the terms \mathbf{s}^{t-1} and \mathbf{s}^{t+1} for t=0 and t=T respectively. This entire system has translational symmetry, since the energy Eq. (4) remains the same if we add the same constant to all ranks at all times, but we can again break this symmetry by setting the mean rank to zero.

Additionally, in contrast to Eq. (5), the ranks at t now depend on both t-1 and t+1, which themselves depend on ranks at adjacent time-steps, so that ranks are affected by interactions in both the past and the future. In computer science, methods like this where the entire history is provided to the algorithm are called *offline*, to distinguish them from *online* approaches that update their results in real time as data becomes available. Thus we refer to this model as Offline Dynamical SpringRank (OFFDSR).

The cost of solving Eq. (5) for a single time-step is the same as static SpringRank with only one additional parameter to be tuned using cross-validation, and there are T such N-dimensional equations to be solved successively. On the other hand, Eq. (6) requires solving a single system of dimension NT, whose operator consists of T blocks, each of dimension $N \times N$. While these two approaches feature numbers of non-zero entries that are fundamentally determined by the number of total edges across all time steps, the cost of solving DSR vs OFFDSR will depend on the particular choice of linear solver [28].

Philosophically, Eqns. (5) and (6) are trying to do two different things. If we are given all the data A^0, A^1, \ldots, A^T and we want to infer retrospectively how each individual's rank changed over time, it makes sense to include both past and future interactions as in (6) so that s_i^t is affected by i's entire history.

In contrast, (5) can be viewed as modeling each individual's perceived rank at the time, based only on the interactions that have occurred so far.

In principle, one could envisage other ways to formally incorporate an explicit dependence on \mathbf{s}^{t-1} into the model, and we provide one example in SI Sec. S3. However, we found that the approaches presented in this Section provide a natural interpretation, result in good prediction performance on both real and synthetic datasets (see Sec. IV) and are computationally scalable.

We close this section with two possible extensions to these models. First, in some settings we might have timestamps t that are not successive integers $0,1,\ldots,T$. In this case, if the time interval between two successive times is Δt , one could scale the spring constant of the self-springs between time-steps as $k/\Delta t$. This corresponds to the fact that if we have Δ identical springs in series, each of which is stretched by $(s^t - s^{t-1})/\Delta$, their total energy is

 $(1/2)(k/\Delta)(s^t - s^{t-1})^2$. The same expression applies if the timestamps are real-valued so that Δ is not an integer.

Second, if we believe that not just the ranks themselves but their rates of change behave smoothly over time, one could add a momentum term to the Hamiltonian which is quadratic in the discrete second derivative of the ranks. Since

$$((s^{t+1} - s^t) - (s^t - s^{t-1}))^2 = (s^{t+1} - 2s^t + s^{t-1})^2$$

= $2(s^t - s^{t-1})^2 + 2(s^{t+1} - s^t)^2 - (s^{t+1} - s^{t-1})^2$,

this is equivalent to adding a repulsive force, i.e., a spring with negative spring constant, between ranks two timesteps apart. Note that the system nevertheless remains convex: this momentum term is positive semidefinite, so adding it to (4) keeps the coupling matrix positive definite except for translational symmetry. Of course, these terms are second-order in time. In the online approach, one would have to determine \mathbf{s}^0 from the static model, \mathbf{s}^1 from the first-order model (5), and then use the model including this momentum term for \mathbf{s}^t for $t \geq 2$. We have not pursued this here, but it may make sense for certain datasets.

A. Moving-window SpringRank

Before the various versions of we test Dynamical SpringRank defined above, we consider a simpler model as a baseline. The simplest way to extend SpringRank to a dynamical context is to apply the static model to the interactions in a series of "windows," where in each window we sum the interactions over a series of consecutive time-steps. For instance, we can compute \mathbf{s}^t for each t by applying the static model to a window of width τ , i.e., replacing A^t with $\sum_{t'=t}^{t+\tau-1} A^{t'}$. Since these windows overlap, the resulting estimates \mathbf{s}^t will be smooth to some extent, even without imposing an explicit dependence between \mathbf{s}^t and \mathbf{s}^{t-1} . We use this method, which we call moving-window SpringRank (mwSR), as a baseline to compare with the dynamical models presented above.

Roughly speaking, a larger τ is like a larger self-spring constant k, since it induces more overlap between windows and thus a stronger correlation between the inferred ranks. However, like a decaying-history approach, mwSR assumes a particular kernel for the importance of past time-steps: namely, that all t' in the window are equally important. In contrast, Dynamical SpringRank infers the importance of past time-steps by coupling \mathbf{s}^t with \mathbf{s}^{t-1} .

However, both models have a free parameter that needs to be tuned, i.e., k and τ . A shorter window τ or smaller spring constant k allows the ranks to respond quickly to new interactions, while a longer window or larger spring constant more tightly couples nearby estimates. This trade-off suggests the existence of an optimal window length $\tau_{\rm opt}$. We tune τ using a cross-validation procedure as explained in SI Sec. S5.

B. Generative Model and Synthetic Data

Analogous to a model presented in [1], we propose a probabilistic generative model for dynamic data. It takes as input the ranks \mathbf{s}^t and generates a sequence of weighted directed networks with adjacency matrix A^t at time t. One can also imagine models that generate the ranks, for instance with a random walk with Gaussian steps whose log-probability is the self-spring Hamiltonian (3), but we treat \mathbf{s}^t as an input since we want the user of this model to have control over how the ground-truth ranks vary with time. For instance, in our experiments below we generate synthetic data where the ranks vary sinusoidally.

The generative model has two real-valued parameters: a signal-to-noise ratio or inverse temperature β , and an overall density of edges c. Given the ranks \mathbf{s}^t , it generates weighted, directed edges between each pair of nodes i,j independently, as follows. The probability $P_{ij}^t(\beta)$ of i "beating" j at time t, giving a directed edge $i \rightarrow j$, is a logistic function as in [1] or the Bradley-Terry-Luce model [12, 13]:

$$P_{ij}^{t}(\beta) = \frac{1}{1 + e^{-2\beta(s_i^t - s_j^t)}}.$$

The number of such edges, which gives the integer weight A_{ij}^t , is then drawn from a Poisson distribution whose mean λ_{ij}^t is $cP_{ij}^t(\beta)$:

$$A_{ij}^t \sim \text{Poi}\left(\lambda_{ij}^t = \frac{c}{1 + e^{-2\beta(s_i^t - s_j^t)}}\right). \tag{7}$$

Since $P_{ij}^t(\beta) + P_{ji}^t(\beta) = 1$, for any pair i, j the total number of interactions $A_{ij}^t + A_{ji}^t$ is Poisson-distributed with mean c. The rank differences $s_i^t - s_j^t$ are used only to choose the directions of these edges. This is equivalent to a model where we define a random multigraph where the number of edges between i and j is Poi(c), and then we choose the direction of each edge independently according to P_{ij}^t .

This is different from the generative model proposed in the static case in [1]. In that model the probability that i and j interact depends on s_i — s_j so that nodes are more likely to interact if their ranks are fairly close. This is consistent with SpringRank's assumption that if i beats j then j is below i, but not too far below it (since the springs have resting length 1). This assumption makes sense for some datasets but not for others. By generating synthetic data without this dependence, our intent is to pose a greater challenge to SpringRank by modeling (for example) round-robin tournaments where every team plays each other.

C. Model Evaluation

Assessing a ranking model on real datasets is not straightforward since we do not know the true values of the underlying ranks. Nevertheless, we may measure the extent to which inferred ranks are accurate in the sense that they can predict the outcome of new observations.

There are several performance metrics that can be used for prediction evaluation. From coarse-grained measures capable of predicting the likely winner to more fine-grained measures that also estimate odds, we consider four main metrics in our experiments, detailed in Sec. S4. We measure prediction performance using a cross-validation protocol where datasets are divided into training and test sets. The training set is used for hyperparameter tuning and parameter estimation while performance is evaluated on the test set. In order to preserve the chronological ordering of the data, the test set contains future observations, i.e., observations that chronologically follow those used in training. Hyperparameters for each method are tuned using grid-search in order to maximize the performance metrics as described in SI Sec. S5.

IV. Results

Having introduced Dynamical SpringRank and its generative counterpart, as well as discussing model selection between the dynamic and static versions of SpringRank, we now illustrate their behavior on synthetic and real data.

We compare prediction performance on held-out test data for DSR and OFFDSR against several state-of-the-art algorithms such as the Elo Rating System (Elo) [21], TrueSkill (TS) [26], "win-loss" decay-history rating (W-L) [24] and Whole-History Rating (WHR) [27]. In addition, we consider two baselines: static SpringRank (SR) [1] and mwSR presented above in Sec. III A. (Note that static SpringRank is the limiting case of mwSR with one window covering the entire dataset.)

A. Performance on Synthetic Data

We first consider synthetic data, generated as described in Sec. IIIB, in which ranks evolve according to periodic ground truth dynamics,

$$s_i^t = b_i \cos(\omega_i t + \phi_i) + c_i \cos(\upsilon_i t + \phi_i), \qquad (8)$$

where b_i, c_i , ω_i , ϕ_i , v_i are parameters randomly chosen for each node from a continuous uniform distribution (see Sec. S6 for details). This results in changes in rankings, and swaps in the order of ranks, reminiscent of real scenarios where teams and players rise and fall.

In order to assess the effect of different network structures, we vary parameters β and c from Eq. (7). We tabulate the results in Table I for varying values of β and fixed c=0.5, and in Table S2 for varying values of c and fixed $\beta=2.0$. We use 50% of the data for training and 4 timesteps for testing, detailed in Sec. S5.

Overall, DSR has the largest number of top performances when considering all metrics (Tables I and S2). Notably, DSR outperforms its offline variant OFFDSR, even though OFFDSR is given the entire history. This implies that using future interactions to retrodict out-of-sample interactions is less accurate than simply using past interactions.

Recall also that DSR is more efficient algorithmically than OFFDSR. Overall, all algorithms perform better for higher values of β (i.e., lower noise).

The model with the second largest number of top performances is WHR, which does well particularly for σ_L , the metric that accounts for the likelihood of the outcomes. Notably, static SpringRank is significantly worse than the other models, illustrating that performance can be negatively affected by choosing a static model in dynamical settings. However, for higher noise levels such as $\beta=0.1$, static SpringRank performs comparably well to the other models. This suggests that when there is less structure in the data, a static algorithm is enough: taking the chronological order of events into account does not improve performance.

As a sanity check of our permutation test for model selection between static and dynamic models, we also considered synthetic datasets generated with static ranks $s_i^t = s_i$. As expected, static SpringRank performs well in comparison to the dynamic algorithms as shown in Table S4.

Finally, we qualitatively investigate the inferred ranking in Fig. 2 for DSR, Elo and W-L where the hierarchy as well as predictive performance is the strongest, as can be seen in Table I when $\beta=2.0$. We notice how the time-scale of the evolution of the ranks is different in all cases, with W-L having frequent and sudden jumps while DSR and Elo are smoother with roughly equal performance. In all cases though, we notice small jumps indicating changes in ranks that deviate from the smoothness in the ground truth. Nevertheless performance is strong for DSR and Elo, who perform roughly equally well, as the behaviors of the individual trajectories resembles that of the ground truth well in both cases.

These synthetic tests suggest that dynamical algorithms capture relevant information when the data has a hierarchical structure and chronological ordering matters (i.e low noise). In these cases, Dynamical SpringRank performs the best according to several metrics. For higher noise levels or static ranks, timestamp information is no longer relevant and static SpringRank performs well.

B. Performance on Real Data

We consider a variety of real datasets of timestamped interactions, as described in Table II. These datasets come from competitions in well-known sports such as soccer, basketball and chess. They are both relevant and relatable sources of information for our experiments.

In soccer, we consider the Italian Serie A and the English Premier League (EPL). The Serie A data is from the period 1993–2016 and contains the results of thousands of games between 47 teams. Similarly, the EPL contains results of thousands of games between 39 teams in the period 2006–2018. In contrast, the NBA dataset contains roughly three times the number of EPL matches from 2010-2018 between 30 teams. All these 3 dataset can be found on *kaggle.com*. Finally, the chess dataset is obtained from matches on *lichess.org*. It contains 298 matches from 2014-2017. In

β	Metric	Elo	OFFDSR	mwSR	DSR	SR	TS	W-L	WHR
0.1	accuracy	0.545	0.544	0.533	0.549	0.540	0.548	0.489	0.549
	agony	1.568	1.596	1.658	1.574	1.646	1.551	1.909	1.578
	σ_a	0.593	0.594	0.576	0.584	0.594	0.593	_	0.592
	$\sigma_{\scriptscriptstyle L}$	-1.426	-1.382	-1.389	-1.378	-1.382	-1.392	-	-1.389
0.5	accuracy	0.700	0.700	0.698	0.700	0.652	0.703	0.384	0.701
	agony	0.881	0.877	0.887	0.877	1.075	0.885	2.416	0.882
	σ_a	0.666	0.647	0.708	0.705	0.635	0.674	-	0.670
	$\sigma_{\scriptscriptstyle L}$	-1.344	-1.286	-1.165	-1.163	-1.263	-1.167	-	-1.152
1.0	accuracy	0.810	0.816	0.810	0.810	0.713	0.808	0.279	0.811
	agony	0.455	0.436	0.429	0.440	0.799	0.458	2.826	0.442
	σ_a	0.771	0.783	0.813	0.813	0.702	0.767	_	0.756
	$\sigma_{\scriptscriptstyle L}$	-1.143	-0.988	-0.848	-0.853	-1.149	-0.863	-	-0.846
1.5	accuracy	0.866	0.862	0.863	0.864	0.752	0.865	0.232	0.863
	agony	0.260	0.269	0.269	0.261	0.655	0.266	3.085	0.270
	σ_a	0.835	0.823	0.863	0.866	0.745	0.825	_	0.815
	$\sigma_{\scriptscriptstyle L}$	-0.883	-0.918	-0.671	-0.670	-1.128	-0.662	-	-0.655
2.0	accuracy	0.898	0.898	0.898	0.903	0.772	0.900	0.195	0.900
	agony	0.172	0.179	0.171	0.163	0.606	0.172	3.229	0.169
	σ_a	0.876	0.847	0.899	0.901	0.769	0.861	_	0.856
	$\sigma_{\scriptscriptstyle L}$	-0.673	-0.844	-0.492	-0.500	-1.088	-0.500	-	-0.492

TABLE I. Results obtained from synthetic data with varying noise levels, represented by β . Each value is the mean of 4 independent realizations of the noisy model. The green highlighted values are the top performances for the considered metric. Notably, some of the values in the same row appear identical but only a single value is highlighted. The reason for this is that the highlighted value is better by less than three decimal places. Table S1 contains the standard error of the above values. σ_a and σ_L cannot be applied to the W-L model, so there are no values for the metrics.

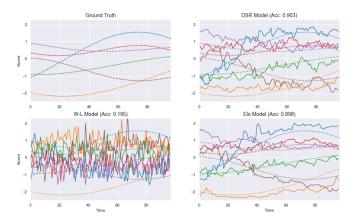


FIG. 2. Evolution of inferred ranks over time on synthetic data. We illustrate the inferred ranks of three models over time: DSR, W-L and Elo. We also illustrate the ground truth of the synthetic ranks over time as a comparison (top left). The synthetic data is generated by setting $\beta=2.0$ and c=0.5. Dashed lines are ground truth ranks.

all cases, A_{ij}^t is the number of times team i (or for chess, player i) beats j in a given time-step t. The definition of a time-step varies from sport to sport (see below).

As with synthetic data, we found that DSR outperforms the other algorithms in terms of the most top performances across our four different metrics (Table III). Elo and WHR are the next best performers: Elo does slightly better in a few cases on the accuracy or agony metric for NBA and chess, and as in the synthetic data WHR does well for the

TABLE II. Descriptions of the real datasets.

Competition	Туре	N_{teams}	N_{games}	T_{steps}
NBA	Basketball	30	9594	218
lichess.org	Chess	96	298	90
Serie A	Soccer	47	5679	397
English Premier League (EPL)	Soccer	39	3396	114

 σ_L metric, the conditional log-likelihood of generating directed edges (outcomes) given their existence.

Dataset	Metric	Elo	OFFDSR	mwSR	DSR	SR	TS	WHR	W-L
NBA	accuracy	0.650	0.642	0.637	0.649	0.607	0.645	0.648	0.564
	agony	2.981	3.048	3.084	2.987	3.568	3.006	2.997	4.070
	σ_a	0.579	0.562	0.639	0.646	0.596	0.584	0.580	_
	$\sigma_{\scriptscriptstyle L}$	-1.426	-1.331	-1.266	-1.256	-1.324	-1.280	-1.255	-
Chess	accuracy	0.526	0.477	0.422	0.513	0.521	0.500	0.481	0.099
	agony	8.253	11.485	9.325	8.318	7.925	7.133	6.946	0.641
	σ_a	0.615	0.580	0.625	0.651	0.581	0.635	0.628	_
	$\sigma_{\scriptscriptstyle L}$	-1.290	-1.341	-1.332	-1.333	-1.550	-1.246	-1.200	-
EPL	accuracy	0.678	0.681	0.669	0.675	0.679	0.672	0.673	0.609
	agony	3.239	4.144	4.141	3.147	3.401	3.797	3.825	5.386
	σ_a	0.595	0.530	0.669	0.675	0.662	0.601	0.598	-
	$\sigma_{\scriptscriptstyle L}$	-1.285	-1.357	-1.208	-1.184	-1.206	-1.211	-1.199	-
Serie A	accuracy	0.655	0.652	0.629	0.653	0.663	0.655	0.653	0.543
	agony	4.296	5.800	6.278	4.041	4.241	5.669	5.653	8.021
	σ_a	0.582	0.530	0.628	0.652	0.647	0.590	0.585	-
	$\sigma_{\scriptscriptstyle L}$	-1.363	-1.357	-1.287	-1.240	-1.269	-1.257	-1.237	-

TABLE III. **Results obtained from real data.** The green highlighted values are the top performances for the considered metric. Notably, some of the values in the same row appear identical but only a single value is highlighted. The reason for this is that the highlighted value is better by less than three decimal places. Table S6 contains the standard error of the above values. σ_a and σ_L cannot be applied to the W-L model hence there are no values for the metrics.

Perhaps surprisingly, static SpringRank performs well on both the Serie A and chess datasets, achieving the highest accuracy on Serie A. For Serie A, this could, in part, be explained by the fact that the dataset has larger time gaps between matches, implying less of a connection between time-steps, hence time may not play as much of a role in the Serie A as it does in the NBA dataset.

For the chess dataset, each time-step represents a day of matches, but match days are not necessarily consecutive. For example, the first day of matches is 2014-03-04 and the second is 2015-11-15. Again, this poses the problem of large gaps in time which could lessen the connection between time-steps.

As such, in both the Serie A and chess datasets, it is understandable that the static version of SpringRank would perform fairly well as time-steps do not influence each other as much as in, for example, the NBA dataset. This is further supported by the closeness in results between the static version of SpringRank and the dynamic models on the soccer and chess datasets. In contrast, the gap of results from the NBA dataset between the aforementioned static and dynamic models is larger. We discuss the influence of time further in Sec. IV C. (The Serie A and chess datasets might also be suitable for the model described above where time

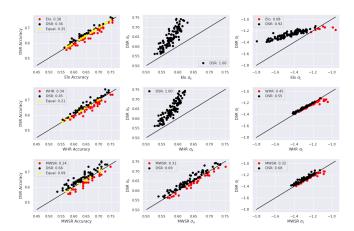


FIG. 3. Fold-by-fold evaluation on the NBA dataset. We compare the predictions of DSR to Elo, WHR and mwSR in relation to the performance metrics σ_a , σ_L and accuracy. The black points above the diagonal represent folds where DSR outperformed its competitors; yellow points indicate equal performance and red points represent DSR loses (where it was outperformed by competitors). Numbers inside the legend are the number of trials that an algorithm outperforms the other in percentage.

intervals between snapshots can vary; we leave this for future work.)

Overall, we observe a fairly broad distribution of values for the various metrics over the cross-validation trials, as there are matches that are more difficult to predict than others. Hence, we take a closer look by analyzing a fold-by-fold performance comparison, where we assess the number of test sets in which one algorithm outperforms the others. We find that DSR performs equal to or better than the other algorithms in most cases on the NBA dataset, and in all cases when compared to Elo and WHR in terms of σ_a (Fig. 3).

We observed qualitative differences of the inferred ranks in Fig. 4 similar to those observed in Fig. 2 for synthetic data. W-L infers ranks that change with a much higher frequency than the others. While smoother, the ranks inferred by TS show more frequent variations than DSR and Elo, which infer similarly behaving ranks.

C. The Relevance of Time

As a final consideration, we turn to a fundamental question: given a dataset of timestamped interactions, does their chronological order matter? If the answer is positive, then we should use a dynamical ranking algorithm to analyze the data. If not, a simpler static algorithm should be enough.

One way to assess whether a given dataset is better modeled by a dynamical or static algorithm is by randomly permuting the order of interactions—but not their outcomes—and thus removing any relationship between ranks and time. If an algorithm performs significantly better on the original data than on the permuted data, this shows that

the order matters and a dynamical model is justified. To be more precise, applying random permutations to the data produces a distribution of any test statistic, including any measure of the performance of an algorithm that predicts which way a given interaction will go (e.g., which of two players will win a chess match, conditioned on the event that they play). If the performance on the original data is far out in the tail of this distribution, we can reject the null hypothesis that the time-steps are simply independent draws from a static model.

We run this permutation test first on synthetic data, confirming as expected that the dynamical model performs significantly better on synthetic data generated with the timevarying model introduced in Sec. III B, provided that the hierarchy itself is sufficiently strong (Fig. S2). However, when the hierarchy is weak (i.e., β is small), the ranks have little relationship to the outcomes, and treating the ranks dynamically is no longer justified by the permutation tests (Fig. S2).

For NBA data, permutation tests show that chronological order matters, and that using a dynamical model significantly improves prediction (Fig. 5). However, for the soccer and chess datasets, we find mixed results depending on the test statistic. For instance, the "agony" (a measure which penalizes the model for interactions $i \rightarrow j$ if $s_i - s_i$ is large) suggests that time-order is relevant, while the accuracy (the fraction of interactions whose direction is correctly predicted) is less sensitive to this information (Fig. S4, Table S7). While the most straightforward explanation is that NBA rankings are more time-varying, while soccer and chess are less so, we also note that there are many more games in a NBA season than in a soccer season, since there are more teams and more frequent games in the NBA, therefore allowing our simple permutation test to reject the null hypothesis more easily with more available data to differentiate time-varying versus static ranks (Table II).

Conclusion

Dynamical SpringRank is a principled extension of the physics-inspired SpringRank model for dynamic hierarchal structures, which lets us infer time-varying ranks from timestamped interactions. By coupling individuals' previous and current ranks, it exploits the chronological ordering of the data to better predict the outcomes of future interactions. It contains a parameter k that can be tuned or learned in order to control the smoothness of the change in ranks, or equivalently the weight given to past ranks.

We constructed two different formulations of Dynamic SpringRank: an online and an offline one, which are given just past ranks and the entire history respectively. The online version performed better and is less computationally expensive. However, both models, similar to the static version, are scalable algorithms that require sparse linear algebra and provide a probabilistic generative model for creating dynamically directed networks with tunable levels of

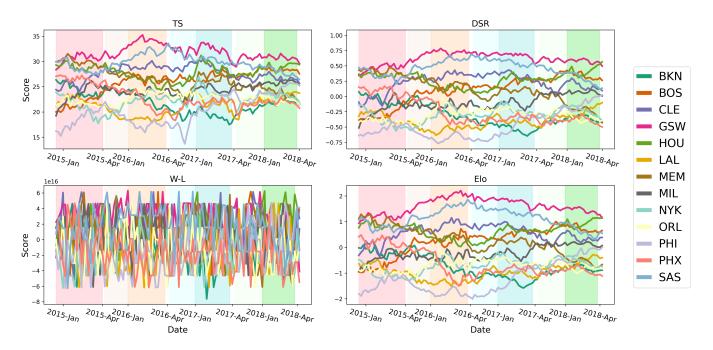


FIG. 4. Evolution over time of predicted ranks for the NBA dataset. We illustrate the predicted ranks of four models over time: TS, DSR, W-L and Elo. We select a subset of 13 teams (as indicated in the legend) to highlight the behaviors of both top and bottom scoring teams. Vertical colored bands break seasons into two periods.

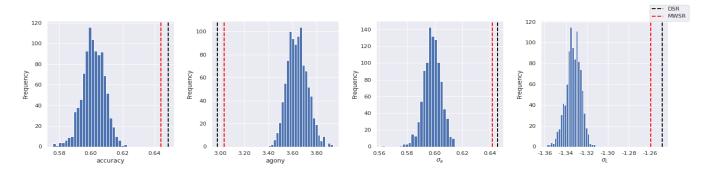


FIG. 5. **Permutation test results on the NBA dataset: chronology matters.** The histogram is generated by 1000 random permutations to the NBA dataset, and measuring the performance of Dynamical SpringRank on these permuted datasets. The black and red dotted lines represent the results of DSR and mwSR respectively on the original, chronologically-ordered NBA dataset — the accuracy is much higher, and the agony much lower, than the vast majority of permuted datasets. This convincingly rejects the null hypothesis that chronological order does not matter, and justifies the use of a dynamical model. In each case the *p*-value is less than 0.001.

hierarchy and sparsity.

We also illustrated that in dynamic settings where time information is important, Dynamical SpringRank is better than its static counterpart. Its ability to predict future outcomes in dynamical settings proved to be similar or better than other state-of-the-art dynamical ranking algorithms for a variety of metrics and datasets, both synthetic and real. An open-source implementation of both *offline* and *online* versions of Dynamical SpringRank is available at https://github.com/cdebacco/DynSpringRank.

For future work, we defined more elaborate models

where the time intervals between interactions can vary, or where a momentum term induces smoothness in the rate at which ranks change over time. Another (perhaps challenging) direction is to couple the rank dynamics with the entities' choices to interact with each other. For instance, one can imagine a model in which animals tend to challenge those immediately above them in the dominance hierarchy, or where new arrivals to a community test themselves against current members in order to find their place, or even three-way interactions where an animal who attacks another is punished by a third [29]. Testing these models would require rich data from biological and social systems.

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Supplemental Information (SI)

S1. Full Derivation Self-Spring Interaction

Calculate the *i*-th component of the gradient:

$$\begin{split} \frac{\partial H^t}{\partial s_i^t} &= \sum_j [A_{ij}^t(s_i - s_j - \ell_0) - A_{ji}^t(s_j - s_i - \ell_0)] + k(s_i^t - s_i^{t-1}) \\ &= \sum_j (A_{ij}^t + A_{ji}^t) s_i^t - \sum_j (A_{ij}^t + A_{ji}^t) s_j^t - \sum_j (A_{ij}^t - A_{ji}^t) \ell_0 + k(s_i^t - s_i^{t-1}) \\ &= (d_i^{out,t} + d_i^{in,t} + k) s_i^t - \sum_j (A_{ij}^t + A_{ji}^t) s_j^t - (d_i^{out,t} - d_i^{in,t}) \ell_0 - k s_i^{t-1}. \end{split}$$

Imposing $\nabla H = 0$ we obtain:

$$(d_i^{out,t} + d_i^{in,t} + k)s_i^t - \sum_i (A_{ij}^t + A_{ji}^t)s_j^t = (d_i^{out,t} - d_i^{in,t})\ell_0 + ks_i^{t-1},$$

which yields:

$$[D^{out,t} + D^{in,t} - (A^t + A^{t,T}) + k\mathbb{I}] \mathbf{s}^{t,*} = [D^{out,t} - D^{in,t}] \ell_0 + k_0 \mathbf{s}^{t-1},$$

as reported in (5) for $\ell_0 = 1$.

S2. Full Derivation of Self-Spring Interaction Over All Time

Calculate the *i*-th component of the gradient:

$$\begin{split} \frac{\partial H_{\text{total}}}{\partial s_{i}^{t}} &= \frac{\partial}{\partial s_{i}^{t}} \left[\sum_{t}^{T} H^{t}(\boldsymbol{s}^{t}, \boldsymbol{s}^{t-1}) \right] \\ &= \frac{\partial}{\partial s_{i}^{t}} \left[\sum_{t}^{T} \sum_{t} A_{ij}^{t} H_{ij}(s_{i}^{t}, s_{j}^{t}) \right] + \frac{\partial}{\partial s_{i}^{t}} \left[\sum_{t}^{T} \sum_{t} k H_{\text{self}}(s_{i}^{t}, s_{i}^{t-1}) \right] \\ &= \sum_{j}^{N} \left[A_{ij}^{t}(s_{i}^{t} - s_{j}^{t} - \ell_{0}) - A_{ji}^{t}(s_{j}^{t} - s_{i}^{t} - \ell_{0}) \right] - k s_{i}^{t-1} + 2 k s_{i}^{t} - k s_{i}^{t+1} \\ &= \sum_{j} (A_{ij}^{t} + A_{ji}^{t}) s_{i}^{t} - \sum_{j} (A_{ij}^{t} + A_{ji}^{t}) s_{j}^{t} - \sum_{j} (A_{ij}^{t} - A_{ji}^{t}) \ell_{0} - k s_{i}^{t-1} + 2 k s_{i}^{t} - k s_{i}^{t+1} \\ &= (d_{i}^{out,t} + d_{i}^{in,t} + k) s_{i}^{t} - \sum_{j} (A_{ij}^{t} + A_{ji}^{t}) s_{j}^{t} - (d_{i}^{out,t} - d_{i}^{in,t}) \ell_{0} - k s_{i}^{t-1} + 2 k s_{i}^{t} - k s_{i}^{t+1} . \end{split}$$

Imposing $\nabla H = 0$ we obtain:

$$(d_i^{out,t} + d_i^{in,t} + 2k)s_i^t - \sum_j (A_{ij}^t + A_{ji}^t)s_j^t = (d_i^{out,t} - d_i^{in,t})\ell_0 + k(s_i^{t-1} + s_i^{t+1}),$$

which yields:

$$[D^{out,t} + D^{in,t} - (A^t + A^{t,\dagger}) + 2k\mathbb{I}]s^{t,*} = [D^{out,t} - D^{in,t}]\ell_0 + k_0(s^{t-1} + s^{t+1}),$$

as reported in (6) for $\ell_0 = 1$.

S3. Dynamic Spring Rest Length

As an alternative to the time-dependency presented in the main text (i.e., through self-springs), we also investigated the introduction of a time-dependent rest length. In this case we assume a dynamic rest length ℓ_{ij}^t for the interaction at time

t between i and j. To enforce a relationship between current and past ranks, we assume ℓ_{ij}^t to be a function of the rank difference $s_i^{t-1} - s_i^{t-1}$ between i and j at time t-1:

$$H_{ij}^{t}(s_{i}^{t},s_{j}^{t}) = \frac{1}{2}(s_{i}^{t}-s_{j}^{t}-l_{ij}^{t})^{2},$$

where

$$l_{ij}^{t} = s_i^{t-1} - s_j^{t-1} + \ell_0. {(S1)}$$

The resultant Hamiltonian for the whole system is,

$$H^{t}(\mathbf{s}^{t},\mathbf{s}^{t-1}) = \sum_{i,j} A_{ij}^{t} H_{ij}^{t} \left(s_{i}^{t}, s_{j}^{t} \right).$$

As opposed to Eq. 4, here we do not have self-interactions. Instead, past ranks appear directly inside the rest lengths. If we define a new variable $z_i^t = s_i^t - s_i^{t-1}$, we obtain the Hamiltonian:

$$H^{t}(\mathbf{z}^{t}) = \sum_{i,i} A_{ij}^{t} H_{ij}^{t} \left(z_{i}^{t}, z_{j}^{t} \right) = \sum_{i,i} \frac{A_{ij}^{t}}{2} \left(z_{i}^{t} - z_{j}^{t} - \ell_{0} \right)^{2}$$

which is the same Hamiltonian used in static SpringRank [1] but as a function of the auxiliary variable \mathbf{z}^t . Thus, we know that the ground state $\mathbf{z}^{t,*}$ will be the solution of the linear system:

$$\left[D^{\text{out}} + D^{\text{in}} - \left(A + A^{\dagger}\right)\right] \mathbf{z}_{t}^{*} = \left[D^{\text{out}} - D^{\text{in}}\right] \ell_{0} \mathbf{1}.$$

The idea is that once $\mathbf{z}^{t,*}$ is obtained by solving this linear system, one can extract the ranks as $s_i^t = z_i^t + s_i^{t-1}$, where s_i^{t-1} is known from the inference of the previous step. Notice that in the extreme case of having only two individuals i, j, i initializing $s_i^0 = s_j^0 = 0$ and i as the constant winner $(A_{ij}^t \geq 0 \text{ and } A_{ji}^t = 0 \,\forall t)$, we would infer $s_i^1 - s_j^1 = \ell_0$ at the first time-step. Then iterating in time yields $\ell_{ij}^t = t\ell_0$. In words, for situations where the hierarchy is strong and time is constant (i.e., a stronger individual always defeats a weaker one at any time-step), the rest length would grow linearly in time. As a consequence, the distance between ranks grows further and further, driving them apart. This is the case in sports, for instance, where teams earn points for each win, distancing them more and more from the losing teams. In other situations, we might want instead a scenario where the difference between ranks becomes a constant value ℓ_0 the more we collect consistent observations in time, i.e., $\forall t, s_i^t - s_j^t = \ell_0$. This can be easily obtain by changing the model's details, like setting a different initial rest length and update in Eq. (S1).

S4. Performance Evaluation

In this section, we discuss the various metrics used in more detail. Accuracy is a coarse-grained measure to evaluate the quality of predictions. It is the fraction of times an observed directed edge points from the higher towards the lower ranked node, i.e., the number of times that a *stronger* (according to our ranking) individual *beats* a weaker one,

$$\operatorname{accuracy} = \frac{1}{M} \sum_{i,j} A_{ij} \Theta(s_i - s_j),$$

where $\Theta(x) = 1$ if x > 0, and 0 otherwise; $M = \sum_{i,j} A_{ij}$. If we call an *upset* an interaction where a lower ranked individual beats someone stronger, then the accuracy is just 1 minus the fraction of upsets. Accuracy does not weigh upsets differently. However, in certain situations making an erroneous prediction involving individuals nearby in rank might be less important than an error involving individuals far in rank. In this case, it is useful to consider the *agony* function [7]. It considers the difference in ordinal ranks as penalities¹. Subsequently, an upset between two nodes close in rank counts much less than an upset between two nodes far rank, based on a parameter d:

agony =
$$\frac{1}{M} \sum_{i,j} A_{ij} \max(0, r_i - r_j)^d$$
,

 $^{^1}$ We use positional ranks instead of the real-valued ranks to avoid scale problems comparing different algorithms

where $r_i \in [0, ..., n-1]$ is the *ordinal* rank of node i (which can naturally be extracted from the real-valued ranks s_i). When d = 0 we recover the standard number of unweighted upsets. The more the rank is informative towards the predicted outcomes, the lower the value of the agony and the less the hierarchy is violated.

Accuracy and agony are metrics for ordinal rankings. For real-valued models such as SpringRank, it is worth considering fine-grained metrics as well. We thus consider in our experiments two other metrics that take into account an estimate of P_{ij} – the probability that i beats j.

First, σ_a is the average probability assigned to the correct direction of an edge:

$$\sigma_a = 1 - \frac{1}{2M} \sum_{ij} |A_{ij} - \overline{A_{ij}} P_{ij}|,$$

where $\overline{A_{ij}} = A_{ij} + A_{ji}$ is the number of interactions between i and j.

Second, σ_L is the conditional log-likelihood of generating the directed edges *given* their existence:

$$\sigma_{L} = \log P\left(A|\bar{A}\right)$$

$$= \sum_{ij} {A_{ij} + A_{ji} \choose A_{ij}} + \log \left[P_{ij}(\beta)^{A_{ij}} \left(1 - P_{ij}(\beta)\right)^{A_{ji}}\right].$$

Notice that we explicitly highlight the dependence of P_{ij} on the (inverse) *temperature* parameter β which control the level of hierarchy in the predictions. For $\beta \to \infty$ the network is fully hierarchical which means that an edge between i and j, with $s_i > s_j$, points from $i \to j$ with $P_{ij} = 1$. In contrast, when $\beta = 0$, the predicted outcomes are completely random with $P_{ij} = P_{ji} = 0.5$.

In general, maximizing σ_a and σ_L requires two distinct values for β that we will denote as $\hat{\beta}_a$ and $\hat{\beta}_L$. Intuitively, the reason is that a single severe mistake where $A_{ij}=1$ but $P_{ij}\approx 0$ reduces the likelihood by a large amount, while only reducing the accuracy by one edge. As a result, predictions using $\hat{\beta}_a$ produce fewer incorrectly oriented edges and achieve a higher σ_a on the test set. On the other hand, predictions using $\hat{\beta}_L$ will produce fewer dramatically incorrect predictions where P_{ij} is very low, and thus achieve higher σ_L on the test set [1]. In other words, a prediction model that maximizes σ_L tends to be more cautious in assigning high probabilities of *success*, even in very unbalanced matches, in order to avoid potential impactful mistakes. In contrast, a model optimizing σ_a can be less conservative, ignoring isolated (even dramatic) mistakes and favoring a good frequency of predictions as close as possible to the real probability.

S5. Cross-Validation and Hyperparameter Tuning

We provide more technical details about the hyperparameter tuning used in the various algorithms and experiments. In all cases, we assume training and test sets have a chronological order, i.e., all matches in the train set happen earlier than those in the test set. Regardless of hyperparameters, all cross-validation folds provide the same exact train/test set to each algorithm for a fair comparison. Importantly, test sets are only used for evaluation.

All results displayed were computed with cross-validation which entailed using 50% of the total data as a train set and 4 time-steps as a test set. This interval was shifted by 1 time-step each fold. Fig S1 demonstrates this process. As a result of cross-validation, there are at most four different values for the same time-step. The reported results are an average of these values.

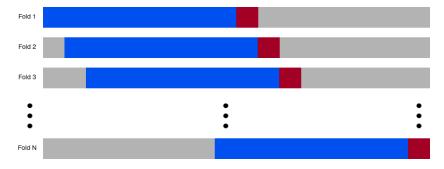


FIG. S1. **Illustration of the cross-validation used in experiments.** The blue bar represents the training set, the red bar is the test and the grey is the total dataset. N is the total number of folds.

As previously mentioned, we used grid-search to perform hyperparameter tuning. For Dynamical SpringRank, grid-search is divided into two steps: first, finding the order of magnitude of k and then progressively finding a more precise value. Refer to Algorithm 1 for the pseudocode of the procedure followed.

```
Algorithm 1: Grid-Search
```

```
    K = {0.001, 0.01, 0.1, 1, 10, 100, 1000}
    for i ← −1, −2, −3 do
    for k in K do
    Find k*, the optimal k that produces best result
    end for
    Update interval K = [k* − 10<sup>i</sup>, k* + 10<sup>i</sup>]
    end for
```

Three algorithms (mwSR, TS and WHR) require an optimal window size, $\tau_{\rm opt}$, for storing data in the training set. We chose this by varying the window size, calculating the average value for each performance metric inside the training set and then choosing the window size corresponding to the best of each of these values. Since the reported results are due to cross-validation, on average the window size of mwSR, TS and WHR on the NBA dataset is $\tau_{\rm opt} = 13, 23, 31$ respectively.

Next, Elo requires a scaling factor k which was determined through a grid-search in the interval [1, 100). WL requires a decaying factor for weighing the data observed and it was set to 0.005. Finally, there are different versions of static SpringRank and we considered the standard version with regularization $\alpha = 0$.

S6. Synthetic Experiments

Periodic Evolution of Synthetic Ranks. We consider a periodic evolution of the ranks generated for synthetic experiments, expressed as Eq. (8). To add detail to the extraction process of the parameters, they were selected from a continuous uniform distribution. The interval of the distribution for parameters was as follows: $b_i, c_i \in [-1, 1)$; $\omega_i, v_i \in [-1, 2)$ and, finally, $\phi_i \in [0, 1)$.

Standard errors. We report standard errors on synthetic experiments where we vary the noise level represented by the parameter β in Table S1. These complement Table I in the main manuscript.

β	Metric	Elo	OFFDSR	mwSR	DSR	SR	TS	W-L	WHR
	accuracy	0.0073	0.0065	0.0067	0.0070	0.0064	0.0068	0.0067	0.0077
0.1	agony	0.0286	0.0269	0.0278	0.0267	0.0331	0.0265	0.03225	0.0284
0.1	σ_a	0.0028	0.0031	0.0044	0.0039	0.0032	0.0029	_	0.0028
	$\sigma_{\scriptscriptstyle L}$	0.0107	0.0008	0.0055	0.0044	0.0019	0.0077	-	0.0066
	accuracy	0.0056	0.0054	0.0057	0.0056	0.0042	0.0051	0.0073	0.0054
0.5	agony	0.0197	0.0196	0.0202	0.0203	0.0168	0.0190	0.0297	0.0185
0.5	σ_a	0.0027	0.0030	0.0051	0.0048	0.0031	0.0029	-	0.0029
	$\sigma_{\scriptscriptstyle L}$	0.0215	0.0035	0.0127	0.0114	0.0061	0.0121	-	0.0110
	accuracy	0.0056	0.0053	0.0048	0.0050	0.0073	0.0047	0.0062	0.0044
1.0	agony	0.0164	0.0156	0.0152	0.0161	0.0297	0.0141	0.0334	0.0142
1.0	σ_a	0.0035	0.0037	0.0044	0.0047	0.0059	0.0030	-	0.0029
	$\sigma_{\scriptscriptstyle L}$	0.0318	0.0079	0.0153	0.0159	0.0254	0.0151	-	0.0135
	accuracy	0.0049	0.0050	0.0050	0.0048	0.0052	0.0049	0.0062	0.0046
1.5	agony	0.0126	0.0132	0.0124	0.0131	0.0210	0.0129	0.0331	0.0118
1.5	σ_a	0.0035	0.0040	0.0046	0.0043	0.0047	0.0037	-	0.0034
	$\sigma_{\scriptscriptstyle L}$	0.0346	0.0080	0.0203	0.0232	0.0263	0.0187	-	0.0148
	accuracy	0.0038	0.0042	0.0036	0.0040	0.0053	0.0047	0.0058	0.0046
2.0	agony	0.0092	0.0095	0.0080	0.0082	0.0206	0.0094	0.0320	0.0089
2.0	σ_a	0.0032	0.0034	0.0035	0.0038	0.0051	0.0034	_	0.0031
	$\sigma_{\scriptscriptstyle L}$	0.0301	0.0073	0.0152	0.0144	0.0302	0.0163	_	0.0147

TABLE S1. Standard error of results from synthetic data with varying noise levels, represent by β

Results for Varying Network Density. In Tables S2 and S3, we show results on synthetic data where we vary the network density represented by the parameter c.

Synthetic Ranks in static scenarios. We consider static ranks $s_i^t = s_i$ generated synthetically using Eq. (8) as a sanity check of our permutation test for model selection between static and dynamic models. Results are shown in Table S4 and Table S5.

c	Metric	Elo	OFFDSR	mwSR	DSR	SR	TS	W-L	WHR
1.0	accuracy	0.905	0.901	0.903	0.904	0.774	0.905	0.156	0.903
	agony	0.161	0.167	0.163	0.165	0.562	0.159	3.423	0.162
	σ_a	0.895	0.892	0.909	0.910	0.778	0.889	_	0.884
	$\sigma_{\scriptscriptstyle L}$	-0.574	-0.705	-0.459	-0.459	-1.054	-0.451	-	-0.450
1.5	accuracy	0.900	0.897	0.902	0.903	0.770	0.902	0.148	0.903
	agony	0.161	0.171	0.159	0.157	0.608	0.162	3.364	0.157
	σ_a	0.899	0.902	0.908	0.909	0.771	0.894	_	0.895
	$\sigma_{\scriptscriptstyle L}$	-0.581	-0.617	-0.462	-0.459	-1.155	-0.460	-	-0.452
2.0	accuracy	0.909	0.905	0.904	0.907	0.768	0.904	0.126	0.904
	agony	0.147	0.156	0.154	0.148	0.601	0.154	3.420	0.153
	σ_a	0.911	0.910	0.915	0.916	0.772	0.907	_	0.902
	$\sigma_{\scriptscriptstyle L}$	-0.579	-0.575	-0.464	-0.453	-1.152	-0.452	_	-0.453
2.5	accuracy	0.904	0.904	0.905	0.906	0.763	0.905	0.124	0.905
	agony	0.159	0.160	0.159	0.155	0.601	0.160	3.417	0.160
	σ_a	0.912	0.913	0.916	0.918	0.773	0.909	_	0.907
	$\sigma_{\scriptscriptstyle L}$	-0.601	-0.650	-0.470	-0.463	-1.172	-0.459	_	-0.458
3.0	accuracy	0.910	0.908	0.908	0.909	0.768	0.909	0.114	0.909
	agony	0.147	0.150	0.152	0.149	0.605	0.152	3.481	0.151
	σ_a	0.921	0.921	0.920	0.921	0.776	0.917	_	0.915
	$\sigma_{\scriptscriptstyle L}$	-0.549	-0.559	-0.455	-0.447	-1.158	-0.438	_	-0.438

TABLE S2. Results obtained from synthetic data with varying density levels, represented by c. Each value is the mean of 4 independent realizations of the model. The green highlighted values are the top performances for the considered metric. Notably, some of the values in the same row appear identical but only a single value is highlighted. This is because the highlighted value is better by less than three decimal places. Table S3 contains the standard error for the above values. σ_a and σ_L cannot be applied to the W-L model hence there are no values for the metrics.

c	Metric	Elo	OFFDSR	mwSR	DSR	SR	TS	W-L	WHR
	accuracy	0.0021	0.0025	0.0023	0.0024	0.0040	0.0024	0.0041	0.0022
1.0	agony	0.0044	0.0055	0.0048	0.0050	0.0138	0.0052	0.0288	0.0051
1.0	σ_a	0.0017	0.0023	0.0020	0.0020	0.0031	0.0021	_	0.0021
	$\sigma_{\scriptscriptstyle L}$	0.0171	0.0087	0.0099	0.0099	0.0200	0.0106	_	0.0095
	accuracy	0.0026	0.0027	0.0025	0.0023	0.0039	0.0026	0.0035	0.0026
1.5	agony	0.0051	0.0056	0.0053	0.0050	0.0180	0.0057	0.0186	0.0056
1.3	σ_a	0.0016	0.0019	0.0021	0.0020	0.0034	0.0016	_	0.0017
	$\sigma_{\scriptscriptstyle L}$	0.0151	0.0053	0.0084	0.0085	0.0296	0.0086	_	0.0085
	accuracy	0.0019	0.0021	0.0020	0.0019	0.0032	0.0022	0.0025	0.0023
2.0	agony	0.0047	0.0044	0.0049	0.0046	0.0130	0.0050	0.0139	0.0052
2.0	σ_a	0.0015	0.0018	0.0019	0.0018	0.0026	0.0016	-	0.0016
	$\sigma_{\scriptscriptstyle L}$	0.0176	0.0079	0.0095	0.0087	0.0273	0.0095	_	0.0089
	accuracy	0.0017	0.0016	0.0016	0.0017	0.0036	0.0017	0.0022	0.0018
2.5	agony	0.0042	0.0040	0.0040	0.0039	0.0140	0.0045	0.0117	0.0046
2.3	σ_a	0.0014	0.0013	0.0015	0.0014	0.0030	0.0014	_	0.0014
	$\sigma_{\scriptscriptstyle L}$	0.0143	0.0065	0.0076	0.0073	0.0285	0.0076	_	0.0072
	accuracy	0.0015	0.0018	0.0016	0.0017	0.0027	0.0017	0.0022	0.0016
3.0	agony	0.0031	0.0037	0.0036	0.0035	0.0146	0.0035	0.0140	0.0037
3.0	σ_a	0.0010	0.0013	0.0014	0.0013	0.0025	0.0010	_	0.0010
	$\sigma_{\scriptscriptstyle L}$	0.0102	0.0046	0.0061	0.0059	0.0278	0.0055	-	0.0054

TABLE S3. Standard error of results from synthetic data with varying density, represented by c.

S7. Real Data Experiments

Standard errors. We report the standard errors of the experiments on the real datasets in Table S6. These complement Table III in the main manuscript.

S8. Null Model Experiments

Synthetic data. We report results of the null model experiments where we permute the chronological order of synthetic dynamic data in Fig. S2 and of synthetic static data in Fig. S3.

Metric	Elo	OFFDSR	mwSR	DSR	SR	TS	W-L	WHR
accuracy	0.715	0.716	0.696	0.720	0.722	0.724	0.360	0.723
agony	0.528	0.518	0.565	0.523	0.498	0.517	1.632	0.515
σ_a	0.666	0.618	0.704	0.733	0.669	0.687	-	0.679
σ_a agony σ_a σ_L	-1.211	-1.324	-1.231	-1.172	-1.148	-1.109	-	-1.111

TABLE S4. Results obtained from synthetic data in a static framework. Performance comparison of the various models on a synthetic dataset where the ranks are fixed along time (static framework). The green highlighted values are the top performances for the considered metric. Table S5 contains the standard error of the above values. σ_a and σ_L cannot be applied to the W-L model, so there are no values for the metrics.

Metric	Elo	OFFDSR	mwSR	DSR	SR	TS	W-L	WHR
accuracy	0.0107	0.0109	0.0109	0.0109	0.0111	0.0101	0.0142	0.0099
$\begin{array}{c} \overline{\text{accuracy}} \\ \text{agony} \\ \sigma_a \\ \sigma_L \end{array}$	0.0243	0.0247	0.0250	0.0261	0.0259	0.0242	0.0391	0.0242
σ_a	0.0059	0.0052	0.0099	0.0095	0.0085	0.0058	-	0.0056
$\sigma_{\scriptscriptstyle L}$	0.0423	0.0049	0.0380	0.0399	0.0223	0.0277	_	0.0251

TABLE S5. Standard error of results from the synthetic data in a static framework.

Dataset	Metric	Elo	OFFDSR	mwSR	DSR	SR	TS	WHR	W-L
	accuracy	0.0048	0.0049	0.0046	0.0048	0.0047	0.0050	0.0047	0.0050
NBA	agony	0.0587	0.0588	0.0621	0.0577	0.0529	0.0576	0.0553	0.0662
NDA	σ_a	0.0021	0.0025	0.0045	0.0045	0.0040	0.0024	0.0022	_
	$\sigma_{\scriptscriptstyle L}$	0.0154	0.0028	0.0075	0.0063	0.0041	0.0097	0.0082	_
	accuracy	0.0353	0.0319	0.0365	0.0324	0.0327	0.0360	0.0358	0.0203
Chess	agony	1.2407	1.8447	1.6864	1.2389	1.2373	1.2945	1.2188	0.3281
Ciless	σ_a	0.0137	0.0134	0.0173	0.0179	0.0123	0.0146	0.0131	_
	$\sigma_{\scriptscriptstyle L}$	0.0773	0.0356	0.0640	0.0753	0.1090	0.0706	0.0507	_
-	accuracy	0.0061	0.0067	0.0073	0.0058	0.0063	0.0064	0.0062	0.0080
EPL	agony	0.0920	0.1292	0.1429	0.0906	0.0935	0.1172	0.1082	0.1611
EPL	σ_a	0.0026	0.0035	0.0068	0.0059	0.0060	0.0030	0.0032	_
	$\sigma_{\scriptscriptstyle L}$	0.0249	0.0038	0.0139	0.0145	0.0112	0.0154	0.0141	_
	accuracy	0.0048	0.0051	0.0051	0.0049	0.0050	0.0050	0.0047	0.0060
Serie A	agony	0.0881	0.1227	0.1282	0.0856	0.0930	0.1252	0.1174	0.1632
serie A	σ_a	0.0020	0.0013	0.0047	0.0044	0.0045	0.0024	0.0022	_
	$\sigma_{\scriptscriptstyle L}$	0.0176	0.0013	0.0102	0.0104	0.0060	0.0107	0.0093	_

TABLE S6. Standard error of results from real data

Real data. We report p-values on the null model experiments on real data in Table S7.

Model	Metric	NBA	Chess	EPL	Serie A
	accuracy	0.0	0.594	0.183	0.359
DSR	agony	0.0	0.006	0.0	0.0
DSK	σ_a	0.0	0.139	0.155	0.282
	$\sigma_{\scriptscriptstyle L}$	0.0	0.711	0.578	0.644
	accuracy	0.0	1.0	0.411	0.999
mwSR	agony	0.0	0.001	0.959	1.0
IIIWSK	σ_a	0.0	0.832	0.340	0.999
	$\sigma_{\scriptscriptstyle L}$	0.0	0.797	0.911	1.0

TABLE S7. **Null model p-value results on real data.** Illustrated are the number of times (as a percentage) that the metric value on the randomized dataset is better than on the chronologically-ordered dataset. Results are calculated over 1000 permutations.

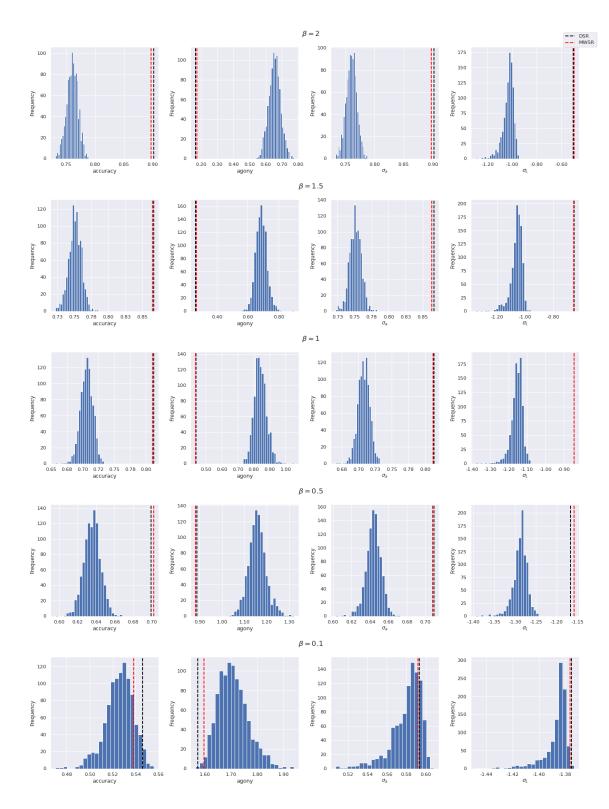


FIG. S2. Null model results on the synthetic dataset with varying levels of density. It is used to determine whether chronology is important. Each entry of the histogram is a different result of DSR on the synthetic dataset where time-steps have been randomly permutated. 1000 permutations were considered. The black and red dotted lines represent the results of DSR and mwSR respectively on the regular, chronologically-ordered dataset.

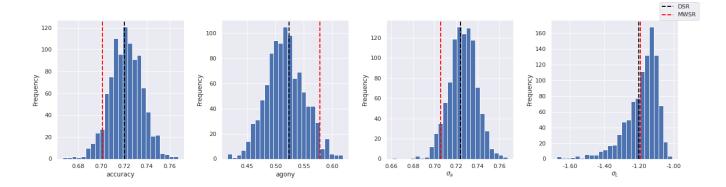


FIG. S3. **Null model results on the synthetic dataset with static ranks.** It is used to determine whether chronology is important. Each entry of the histogram is a different result of DSR on the synthetic dataset where time-steps have been randomly permutated. 1000 permutations were considered. The black and red dotted lines represent the results of DSR and mwSR respectively on the regular, chronologically-ordered dataset.

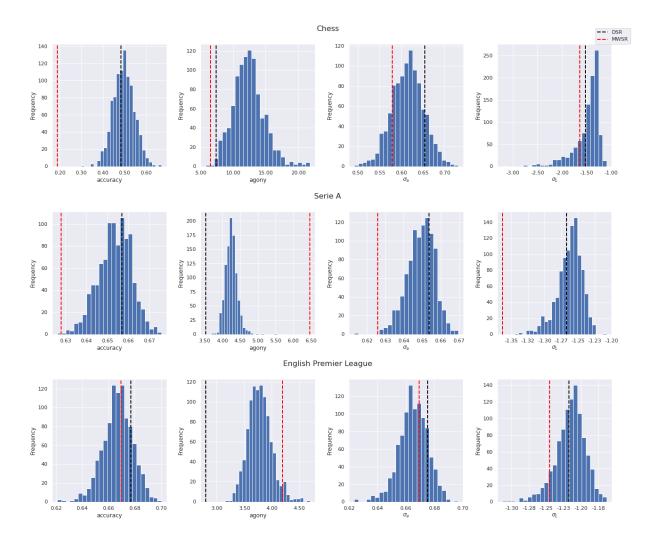


FIG. S4. **Null model results on the chess, EPL and Serie A datasets.** It is used to determine whether chronology is important. Each entry of the histogram is a different result of DSR on the aforementioned datasets, where time-steps have been randomly permutated. 1000 permutations were considered. The black and red dotted lines represent the results of DSR and mwSR respectively on the regular, chronologically-ordered datasets.